Exact results for the generating function of directed column-convex animals on the square lattice

1994 J. Phys. A: Math. Gen. 27 4359

(http://iopscience.iop.org/0305-4470/27/13/012)

J. Phys. A: Math. Gen. 27 (1994) 4359–4367. Printed in the UK

Exact results for the generating function of directed column-convex animals on the square lattice

G S Joyce† and A J Guttman‡§

† Wheatstone Physics Laboratory, King's College, University of London, Strand, London WC2R 2LS, UK

‡ LaBRI, Département d'Informatique, Université Bordeaux I, 351 Cours de la Libération, 33405 Talence Cedex, France

Received 18 October 1993, in final form 15 March 1994

Abstract. The mathematical properties of the generating function $S(x)$ for the number s_n of directed column-convex lattice animals on a square lattice with a given directed-site perimeter n are investigated. In particular, it is shown that $S(x)$ can be expressed exactly in terms of algebraic hypergeometric functions. A detailed investigation of the asymptotic behaviour of s_n as $n \to \infty$ is carried out by applying the Darboux method to the hypergeometric formula for $S(x)$. It is also demonstrated that s_n satisfies a four-term recurrence relation. Finally, it is noted that the techniques used to analyse the lattice animal generating function $S(x)$ can be applied to *any* other generating function which satisfies a cubic algebraic equation.

1. Introduction

The enumeration problem for directed lattice animals, which was first investigated some time ago (Redner and Yang 1982, Dhar *et al* 1982, Day and Lubensky 1982, Stanley *et al* 1982), has continued to give rise to some extremely interesting and often surprising results. In particular, it was shown (Dhar 1982) that the directed-site animal problem on square and triangular lattices is equivalent to the hard-square lattice-gas model with anisotropic next-nearest-neighbour interactions at the disorder point (Baxter 1980, Baxter and Tsang 1980, Baxter and Pearce 1982). This connection was then used by Dhar (1982) to derive exact formulae for the generating functions of directed-site animals on square and triangular lattices. In a similar manner, Dhar (1983) also obtained further exact results for a directed-site animal problem on a three-dimensional simple-cubic lattice (see Joyce 1989). More generally, it has been established by Cardy (1982) that the critical exponents of directed animals on a D-dimensional lattice are related to those for the Lee–Yang edge singularity in the $(D-1)$-dimensional Ising model (Fisher 1978). The application of the well known expression for the free energy of the one-dimensional Ising model to this result enabled Cardy to determine the exact values for the two-dimensional critical exponents of directed animals. There has also been some interest in the enumeration problem for directed and partially directed *compact* lattice animals in two dimensions (Bhat *et al* 1986, 1987, 1988, Privman and Forgacs 1987).

The main aim in this paper is to analyse the recent results of Delest and Dulucq (1993) on the enumeration of directed *column-convex* (DCC) lattice animals with a given site perimeter on a two-dimensional square lattice. We shall begin by giving a precise definition of a DCC

§ Permanent address: Department of Mathematics, The University of Melbourne, Parkville, Vic. 3052, Australia.

0305-4470/94/134359+09$19.50

lattice animal. A lattice *animal* is a set $\mathcal{A}$ of lattice sites such that any pair of sites belonging to $\mathcal{A}$ can be connected by at least one continuous path consisting of nearest-neighbour (NN) edges which only join sites of $\mathcal{A}$. We define a *directed* lattice animal $\mathcal{A}_D$ to be one that has a source point $(0, 0)$ with the property that all other sites in the animal can be reached from $(0, 0)$ by at least one path which is made up entirely of NN steps in either the $+x$ or $+y$ direction and only passes through sites in $\mathcal{A}_D$. A DCC animal $\mathcal{A}_{DCC}$ has the further property that every vertical column of sites in the animal must form an unbroken chain of sites linked by single NN edges. Finally, we define the *site perimeter* of a lattice animal $\mathcal{A}_{DCC}$ to be the number of sites outside $\mathcal{A}_{DCC}$ which can be reached from the boundary sites of the animal by making a single NN step in either the $+x$ or $+y$ direction.

Delest and Dulucq (1993) have proved that the number s_n of DCC lattice animals on the square lattice with a site perimeter n has a generating function

$$S(x) = \sum_{n=0}^{\infty} s_n x^n = x^2 + 3x^3 + 12x^4 + 54x^5 + 260x^6 + 1310x^7 + \dots \quad (1.1)$$

which is a solution of the algebraic equation

$$S^3 + 3(x-1)S^2 + (x-1)(3x-1)S + x^2(x-1) = 0. \quad (1.2)$$

They found that this result can be simplified by introducing the modified generating function

$$T(x) = -1 + x + S(x) \quad (1.3)$$

which satisfies the reduced equation

$$T^3 - 2(1-x)T - (1-x) = 0. \quad (1.4)$$

It is also convenient to express (1.4) in the alternative form

$$\left(\frac{4z}{27}\right) y^3 - y - 1 = 0 \quad (1.5)$$

where

$$y = 2T \quad (1.6)$$

$$z = \tfrac{27}{32}(1-x)^{-1}. \quad (1.7)$$

In section 2 of this paper we shall prove that all the branches of the algebraic function $y = y(z)$ are solutions of the standard hypergeometric differential equation of second order (Erdélyi *et al* 1953). This basic result is then used to determine the analytic properties of the function $y(z)$. In section 3, a four-term recurrence relation for s_n is derived and an analytic continuation formula for the generating function $S(x)$ is obtained which is valid in the neighbourhood of the dominant singularity $x_c = \frac{5}{32}$. Next, the detailed asymptotic behaviour of s_n as $n \to \infty$ is investigated by applying the Darboux method (see Wong 1989) to the singular part of this analytic continuation. Finally, we give an algebraic closed-form expression for the generating function $S(x)$.

2. Analytic properties of the algebraic function $y(z)$

In order to determine the properties of $y(z)$, we first apply the transformations $z = -27v^2/4$ and $y = -v^{-1}Y$ to (1.5). This procedure gives

$$Y(v)\left[1 + Y(v)^2\right] = v. \tag{2.1}$$

The relation (2.1) occurs in the mean-field theory of a ferromagnet as the dimensionless scaling-law form for the critical equation of state above the critical temperature T_c (Widom 1965, Domb and Hunter 1965, Griffiths 1967, Fisher 1967). Joyce (1972) has shown that (2.1) is also associated with the critical equation of state for the spherical model on a D-dimensional lattice with $D > 4$. In the same work, the Lagrange inversion formula (see Whittaker and Watson 1965) was used to express the physically acceptable solution $Y_+(v)$ of (2.1) in terms of the hypergeometric function. From these results it is reasonable to suppose that *any* solution of equation (1.5) will also satisfy the hypergeometric differential equation (Erdélyi *et al* 1953)

$$\mathcal{L}[y] \equiv \left\{z(1-z)\mathrm{D}_z^2 + [c - (a+b+1)z]\mathrm{D}_z - ab\right\}y(z) = 0 \tag{2.2}$$

where $\mathrm{D}_z \equiv \mathrm{d}/\mathrm{d}z$; $a = \frac{1}{3}$, $b = \frac{2}{3}$ and $c = \frac{3}{2}$.

We can prove this conjecture by differentiating (1.5) twice with respect to z. After some algebraic manipulations this procedure yields

$$y'(z) = \tfrac{4}{3}y^3\left(9 - 4zy^2\right)^{-1} \tag{2.3}$$

$$y''(z) = \tfrac{32}{9}y^5\left(27 - 8zy^2\right)\left(9 - 4zy^2\right)^{-3}. \tag{2.4}$$

If formulae (2.3) and (2.4) are substituted into (2.2) we find that

$$\mathcal{L}[y] = 162y\left[\left(\frac{4z}{27}\right)y^3 - y - 1\right]\left[\left(\frac{4z}{27}\right)y^3 - y + 1\right]\left(9 - 4zy^2\right)^{-3}. \tag{2.5}$$

The application of (1.5) to this expression gives $\mathcal{L}[y] = 0$ and the conjecture is verified.

It follows from the hypergeometric equation (2.2) that in the neighbourhood of $z = 0$ the solutions of (1.5) can all be written as

$$y(z) = A_0 z^{-\frac{1}{2}} F\left(-\tfrac{1}{6}, \tfrac{1}{6}; \tfrac{1}{2}; z\right) + B_0 F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; z\right) \tag{2.6}$$

where A_0 and B_0 are constants. We can determine the possible values of the constants A_0 and B_0 by first expanding (2.6) in the form

$$y(z) = A_0 z^{-\frac{1}{2}} + B_0 z^0 + \mathrm{O}(z^{\frac{1}{2}}). \tag{2.7}$$

If we now substitute (2.7) in (1.5) and equate the coefficients of $z^{-\frac{1}{2}}$ and z^0 to zero we find that $\{A_0 = \pm\frac{3\sqrt{3}}{2}, B_0 = \frac{1}{2}\}$ and $\{A_0 = 0, B_0 = -1\}$. In this manner we see that the solutions of (1.5) have the representations

$$y_\pm(z) = \tfrac{1}{2}\left[F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; z\right) \pm (27/z)^{\frac{1}{2}} F\left(-\tfrac{1}{6}, \tfrac{1}{6}; \tfrac{1}{2}; z\right)\right] \tag{2.8}$$

$$y_0(z) = -F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; z\right) \tag{2.9}$$

where $|z| \leqslant 1$.

A similar analysis can be carried out in the neighbourhood of the regular singular point $z = 1$. In this case we can use the linear superposition (Erdélyi *et al* 1953)

$$y(z) = A_1 z^{-\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; 1 - z^{-1}\right) + B_1 z^{-\frac{1}{3}} \left(z^{-1} - 1\right)^{\frac{1}{2}} F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; 1 - z^{-1}\right) \tag{2.10}$$

where A_1 and B_1 are constants. It is found that the constants A_1 and B_1 can take the the values $\{A_1 = 3, B_1 = 0\}$ and $\{A_1 = -\frac{3}{2}, B_1 = \pm\frac{\sqrt{3}}{2}\}$. From these results we obtain the following analytic continuation formulae for the three branches of $y(z)$:

$$y_+(z) = 3z^{-\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; 1 - z^{-1}\right) \tag{2.11}$$

$$y_-(z) = -\left(\tfrac{3}{2}\right) z^{-\frac{1}{3}} \left[F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; 1 - z^{-1}\right) + \tfrac{1}{\sqrt{3}} \left(z^{-1} - 1\right)^{\frac{1}{2}} F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; 1 - z^{-1}\right)\right] \tag{2.12}$$

$$y_0(z) = -\left(\tfrac{3}{2}\right) z^{-\frac{1}{3}} \left[F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; 1 - z^{-1}\right) - \tfrac{1}{\sqrt{3}} \left(z^{-1} - 1\right)^{\frac{1}{2}} F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; 1 - z^{-1}\right)\right]. \tag{2.13}$$

The hypergeometric series in equations (2.11)–(2.13) are absolutely convergent provided that $\mathrm{Re}(z) \geqslant \frac{1}{2}$.

It is also possible to investigate the behaviour of the algebraic function $y(z)$ in the neighbourhood of the point $z = \infty$ by using the linear superposition (Erdélyi *et al* 1953)

$$y(z) = A_2 z^{-\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{2}{3}; z^{-1}\right) + B_2 z^{-\frac{2}{3}} F\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{4}{3}; z^{-1}\right) \tag{2.14}$$

where A_2 and B_2 are constants. We find that the allowed values for these constants are $\{A_2 = 3(2)^{-\frac{2}{3}}, B_2 = 3(2)^{-\frac{4}{3}}\}$ and $\{A_2 = 3(2)^{-\frac{2}{3}} \mathrm{e}^{\pm 2\pi \mathrm{i}/3}, B_2 = 3(2)^{-\frac{4}{3}} \mathrm{e}^{\mp 2\pi \mathrm{i}/3}\}$. Hence, we obtain the further analytic continuation formulae

$$y_+(z) = 3(4z)^{-\frac{2}{3}} \left[F\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{4}{3}; z^{-1}\right) + (4z)^{\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{2}{3}; z^{-1}\right)\right] \tag{2.15}$$

$$y_-(z) = 3(4z)^{-\frac{2}{3}} \left[\mathrm{e}^{-2\pi \mathrm{i}/3} F\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{4}{3}; z^{-1}\right) + \mathrm{e}^{+2\pi \mathrm{i}/3} (4z)^{\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{2}{3}; z^{-1}\right)\right] \tag{2.16}$$

$$y_0(z) = 3(4z)^{-\frac{2}{3}} \left[\mathrm{e}^{+2\pi \mathrm{i}/3} F\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{4}{3}; z^{-1}\right) + \mathrm{e}^{-2\pi \mathrm{i}/3} (4z)^{\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{2}{3}; z^{-1}\right)\right] \tag{2.17}$$

where $|z| \geqslant 1$.

Finally, we derive closed-form expressions for $y(z)$ by applying the standard Cardan formula to the cubic equation (1.5). In this manner we obtain

$$y_+(z) = 3(4z)^{-\frac{1}{3}} \left\{\left[\tfrac{1}{2} - \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\} \tag{2.18}$$

$$y_-(z) = 3(4z)^{-\frac{1}{3}} \left\{\mathrm{e}^{-2\pi \mathrm{i}/3} \left[\tfrac{1}{2} - \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \mathrm{e}^{+2\pi \mathrm{i}/3} \left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\} \tag{2.19}$$

$$y_0(z) = 3(4z)^{-\frac{1}{3}} \left\{\mathrm{e}^{+2\pi \mathrm{i}/3} \left[\tfrac{1}{2} - \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \mathrm{e}^{-2\pi \mathrm{i}/3} \left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 - z^{-1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}. \tag{2.20}$$

When $0 < z < 1$, the results (2.18)–(2.20) are rather inconvenient to use because they express the *real*-valued functions $y_\pm(z)$ and $y_0(z)$ in terms of cube roots of *complex* numbers which are, in general, *irreducible*. It should be noted that the hypergeometric representations for $y_\pm(z)$ and $y_0(z)$ given earlier do not suffer from this disadvantage. A comparison between (2.15)–(2.17) and (2.18)–(2.20), respectively, leads to the well known identities

$$F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{2}{3}; w\right) = \left[\tfrac{1}{2} + \tfrac{1}{2}(1-w)^{\frac{1}{2}}\right]^{\frac{1}{3}} \tag{2.21}$$

$$F\left(\tfrac{1}{6}, \tfrac{2}{3}; \tfrac{4}{3}; w\right) = (4/w)^{\frac{1}{3}}\left[\tfrac{1}{2} - \tfrac{1}{2}(1-w)^{\frac{1}{2}}\right]^{\frac{1}{3}} \tag{2.22}$$

where $|w| \leqslant 1$. Further relations can be obtained by making comparisons with the other hypergeometric formulae for $y_\pm(z)$ and $y_0(z)$.

3. Generating function for DCC animals with given site perimeter

In this section we shall use the properties of the algebraic function $y(z)$ to investigate the enumeration problem for DCC lattice animals.

3.1. Basic results

We follow the work of Delest and Dulucq (1993) and consider the function

$$T(x) = -1 + x + S(x) \tag{3.1}$$

where

$$S(x) = \sum_{n=0}^{\infty} s_n x^n \tag{3.2}$$

is the generating function for the number s_n of DCC lattice animals which have a directed-site perimeter n. It is readily seen from (1.6) and (1.7) that a differential equation for $T(x)$ can be derived from the hypergeometric equation (2.2) with $a = \frac{1}{3}$, $b = \frac{2}{3}$ and $c = \frac{3}{2}$ by making the changes of variable $y \mapsto 2T$ and $z \mapsto \frac{27}{32}(1-x)^{-1}$. Hence, we obtain

$$\left[(5-32x)(1-x)^2\,\mathrm{D}_x^2 - 16(1-x)^2\,\mathrm{D}_x - 6\right]T(x) = 0 \tag{3.3}$$

where $\mathrm{D}_x \equiv \mathrm{d}/\mathrm{d}x$. This differential equation has regular singular points at $x = 1$, $x_c = \frac{5}{32}$ and $x = \infty$. If we solve (3.3) as a Maclaurin series about the ordinary point $x = 0$ we find that the coefficient s_n in the expansion (3.2) must satisfy the four-term recurrence relation

$$5n(n+1)s_{n+1} - 2n(21n-13)s_n + (3n-5)(23n-20)s_{n-1} - 16(n-2)(2n-5)s_{n-2} = 0 \tag{3.4}$$

where $n \geqslant 4$, with the initial conditions $s_2 = 1$, $s_3 = 3$ and $s_4 = 12$. (The recurrence relation is not valid for $n = 1, 2, 3$ because the coefficient of x^m in the power series expansion for $T(x)$ is only equal to s_m for $m \geqslant 2$.)

The behaviour of $S(x)$ in the neighbourhood of the dominant singularity at $x_c = \frac{5}{32}$ can be established by applying the transformations $y \mapsto 2T$ and $z \mapsto \frac{27}{32}(1-x)^{-1}$ to the analytic continuation formula (2.12) for the branch $y_-(z)$. This procedure gives the basic result

$$S(x) = (1-x) - \left(\tfrac{1-x}{2}\right)^{\frac{1}{3}} \left[F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; \tfrac{-5+32x}{27}\right) + \tfrac{1}{9}(5-32x)^{\frac{1}{2}} F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; \tfrac{-5+32x}{27}\right)\right]. \tag{3.5}$$

It is also useful to express (3.5) in the alternative simplified form

$$S(x) = \psi_1(\epsilon) + \epsilon^{\frac{1}{2}} \psi_2(\epsilon) \tag{3.6}$$

where

$$\psi_1(\epsilon) = S(x_c)\left[9(1+\epsilon) - 8(1+\epsilon)^{\frac{1}{3}} F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; -\epsilon\right)\right] \tag{3.7}$$

$$\psi_2(\epsilon) = -\tfrac{\sqrt{3}}{4}(1+\epsilon)^{\frac{1}{3}} F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; -\epsilon\right) \tag{3.8}$$

$$\epsilon = \tfrac{5}{27}\left(1 - \tfrac{x}{x_c}\right) \tag{3.9}$$

and $S(x_c) = \frac{3}{32}$.

3.2. Asymptotic behaviour of s_n as $n \to \infty$

We shall now determine the asymptotic behaviour of s_n as $n \to \infty$ by applying the method of Darboux (see Wong 1989) to (3.6). In the first stage of the analysis, the function $\psi_2(\epsilon)$ in the singular part of (3.6) is expanded as a Maclaurin series in the form

$$\psi_2(\epsilon) = -\tfrac{\sqrt{3}}{4} \sum_{k=0}^{\infty} (-1)^k f_k \epsilon^k \tag{3.10}$$

where $|\epsilon| \leqslant 1$,

$$f_k = \frac{\left(\frac{1}{3}\right)_k \left(\frac{5}{6}\right)_k}{\left(\frac{3}{2}\right)_k k!} \, {}_3F_2\left(-k, -\tfrac{1}{2}-k, -\tfrac{1}{3}; \tfrac{1}{6}-k, \tfrac{2}{3}-k; 1\right) \tag{3.11}$$

and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ denotes a generalized hypergeometric function (Erdélyi *et al* 1953). It can be shown from the differential equation (3.3) that the coefficient f_k also satisfies the recurrence relation

$$9(k+1)(2k+3) f_{k+1} - 2(3k+2)(6k-1) f_k + 9(k-1)(2k-1) f_{k-1} = 0 \tag{3.12}$$

where $k \geqslant 0$, with the initial conditions $f_0 \equiv 1$ and $f_{-1} \equiv 0$. The behaviour of f_k for large k is given by the asymptotic formula

$$f_k \sim -\left(\tfrac{\sqrt{3}}{\pi}\right) \Gamma\left(\tfrac{1}{3}\right) 2^{-\frac{5}{3}} k^{-\frac{4}{3}} \tag{3.13}$$

as $k \to \infty$.

Next, we substitute (3.10) in the singular part of (3.6) and define the Mth Darboux approximant as

$$S[M, x] \equiv -\tfrac{\sqrt{5}}{12} \sum_{k=0}^{M} f_k \left(-\tfrac{5}{27}\right)^k \left(1 - \tfrac{x}{x_c}\right)^{k+\frac{1}{2}}. \tag{3.14}$$

If the binomial theorem is used to expand $S[M, x]$ as a Maclaurin series, we obtain the asymptotic representation

$$s_n \sim s_n[M] \tag{3.15}$$

as $n \to \infty$, where $M = 0, 1, 2, \ldots$, and

$$s_n[M] = (-1)^{n+1} \tfrac{\sqrt{5}}{12} x_c^{-n} \sum_{k=0}^{M} f_k \left(-\tfrac{5}{27}\right)^k \binom{k+\frac{1}{2}}{n}. \tag{3.16}$$

We can prove, using the ratio test and (3.13), that the infinite sequence of approximants $\{s_n[M]; M = 0, 1, 2, \ldots\}$ *converges* to a finite limit $s_n[\infty]$. It should be stressed, however, that this unusual feature does *not* imply that $s_n = s_n[\infty]$! For example, when $n = 12$ we find that

$$s_{12}[\infty] = 6\,141\,763.994\,018\ldots$$

while the exact value is $s_{12} = 6\,141\,764$. The *most accurate* approximation for s_{12} is given by

$$s_{12}[18] = 6\,141\,763.998\,159\ldots.$$

If the standard asymptotic expansion for the binomial coefficient (Luke 1969) is substituted in the approximant (3.16) with $M = \infty$ we obtain

$$s_n \sim \tfrac{1}{24} \left(\tfrac{5}{\pi}\right)^{\frac{1}{2}} n^{-\frac{3}{2}} x_c^{-n} \sum_{m=0}^{\infty} g_m n^{-m} \tag{3.17}$$

as $n \to \infty$, where

$$g_m = \left(\tfrac{3}{2}\right)_m \sum_{k=0}^{m} f_k \left(\tfrac{5}{27}\right)^k [(m-k)!]^{-1} B_{m-k}^{(-k-\frac{1}{2})}(0) \tag{3.18}$$

and $B_n^{(a)}(x)$ denotes a generalized Bernoulli polynomial. The values of the first few coefficients g_m are

$$g_0 = 1 \qquad g_1 = \tfrac{649}{1944} \qquad g_2 = \tfrac{813\,025}{7\,558\,272} \qquad g_3 = -\tfrac{3\,181\,261\,895}{44\,079\,842\,304}.$$

One would expect the asymptotic expansion in (3.17) to be a *divergent* series.

3.3. *Closed-form expressions for $S(x)$*

We have seen that the basic formula (3.5) is particularly useful for investigating the asymptotic behaviour of s_n as $n \to \infty$. However, it is also possible, *at least in principle*, to expand the hypergeometric functions in (3.5) as Maclaurin series about $x = 0$. In this manner, we can also generate, from (3.5), the exact values of $\{s_n; n = 0, 1, 2, \ldots\}$.

In practice the direct expansion of (3.5) about $x = 0$ is most easily carried out by first applying the relations (Oberhettinger 1965)

$$F\left(-\tfrac{1}{6}, \tfrac{1}{3}; \tfrac{1}{2}; w^2\right) = \tfrac{1}{2}\left[(1+w)^{\frac{1}{3}} + (1-w)^{\frac{1}{3}}\right] \tag{3.19}$$

$$F\left(\tfrac{1}{3}, \tfrac{5}{6}; \tfrac{3}{2}; w^2\right) = \tfrac{3}{2} w^{-1} \left[(1+w)^{\frac{1}{3}} - (1-w)^{\frac{1}{3}}\right]. \tag{3.20}$$

Hence, we obtain the algebraic closed-form expression

$$S(x) = (1-x) - \tfrac{1}{2}(1-x)^{\frac{1}{3}} \left\{[\Lambda_+(x) + \Lambda_-(x)] - \mathrm{i}\sqrt{3}[\Lambda_+(x) - \Lambda_-(x)]\right\} \tag{3.21}$$

where

$$\Lambda_\pm(x) = \left[\tfrac{1}{2} \pm \tfrac{\mathrm{i}}{2}\left(\tfrac{5-32x}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \tag{3.22}$$

The formula (3.21) represents the generating function series (3.2) for $|x| \leqslant x_c$ and also gives the analytic continuation of the series throughout the x plane, provided that a cut is made along the positive real axis from $x = x_c$ to $x = \infty$. An alternative procedure for deriving (3.21) is to apply the transformations $y_- \mapsto 2T$ and $z \mapsto \frac{27}{32}(1-x)^{-1}$ to the formula (2.19) for $y_-(z)$. This procedure yields

$$S(x) = (1-x) + (1-x)^{\frac{1}{3}}\left[\mathrm{e}^{+2\pi\mathrm{i}/3}\Lambda_+(x) + \mathrm{e}^{-2\pi\mathrm{i}/3}\Lambda_-(x)\right] \tag{3.23}$$

which is equivalent to (3.21).

Next, we write (3.22) in the form

$$\Lambda_\pm(x) = \left(\tfrac{1}{2} \pm \tfrac{\mathrm{i}}{2}\sqrt{\tfrac{5}{27}}\right)^{\frac{1}{3}} \left\{1 + \tfrac{\sqrt{5}}{32}\left(\sqrt{5} \pm \mathrm{i}3\sqrt{3}\right)\left[\left(1 - \tfrac{32x}{5}\right)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{3}} \tag{3.24}$$

and note the identity

$$\left(\tfrac{1}{2} \pm \tfrac{\mathrm{i}}{2}\sqrt{\tfrac{5}{27}}\right)^{\frac{1}{3}} = \tfrac{1}{4}\left[\left(1 + \sqrt{5}\right) \pm \tfrac{\mathrm{i}}{\sqrt{3}}\left(3 - \sqrt{5}\right)\right]. \tag{3.25}$$

From these results we can derive the expansion

$$\begin{aligned}\tfrac{1}{2}[\Lambda_+(x) + \Lambda_-(x)] - \tfrac{\mathrm{i}\sqrt{3}}{2}[\Lambda_+(x) - \Lambda_-(x)] &= 1 - \tfrac{2}{3}x - \tfrac{10}{9}x^2 - \tfrac{274}{81}x^3 - \tfrac{3\,220}{243}x^4 \\ &\quad - \tfrac{42\,908}{729}x^5 - \tfrac{1\,845\,892}{6\,561}x^6 - \tfrac{27\,778\,780}{19\,683}x^7 - \dots\end{aligned} \tag{3.26}$$

where $|x| \leqslant x_c$. Finally, the substitution of (3.26) into (3.21) leads to agreement with the generating function series (1.1).

4. Concluding remarks

We have seen that the analysis of the algebraic function $y(z)$ given in section 2 enables us to investigate the detailed properties of the generating function $S(x)$. It should, however, be pointed out that the function $y(z)$ can also be used to study the properties of any function $\Omega(\omega)$ which satisfies a *general* cubic equation of the form

$$\Omega^3 + 3a_1(\omega)\Omega^2 + 3a_2(\omega)\Omega + a_3(\omega) = 0 \tag{4.1}$$

where $\{a_j(\omega);\ j = 1, 2, 3\}$ are functions of the independent variable ω.

To establish this connection, we first make the substitution $\Omega = \Theta - a_1(\omega)$ in (4.1). This procedure yields the reduced equation

$$\Theta^3 + p(\omega)\Theta + q(\omega) = 0 \tag{4.2}$$

where

$$p(\omega) = 3\left[a_2(\omega) - a_1^2(\omega)\right] \tag{4.3}$$

$$q(\omega) = a_3(\omega) - 3a_1(\omega)a_2(\omega) + 2a_1^3(\omega). \tag{4.4}$$

If the further transformations

$$\Theta = [q(\omega)/p(\omega)]y \tag{4.5}$$

$$z = -\tfrac{27}{4}\left[q^2(\omega)/p^3(\omega)\right] \tag{4.6}$$

are now applied to (4.2) we obtain the required equation (1.5) for the algebraic function $y(z)$.

Acknowledgments

AJG would like to thank Maylis Delest for bringing the problem of DCC lattice animals to his attention, for many helpful discussions and for her generous hospitality while visiting the Laboratoire Bordelais de Récherche en Informatique at the Université Bordeaux I. We are also extremely grateful to one of the referees and Dr I J Zucker for their constructive comments on the first version of this paper. Financial support from the Australian Research Council is acknowledged.

References

Baxter R J 1980 *J. Phys. A: Math. Gen.* **13** L61–70
Baxter R J and Pearce P A 1982 *J. Phys. A: Math. Gen.* **15** 897–910
Baxter R J and Tsang S K 1980 *J. Phys. A: Math. Gen.* **13** 1023–30
Bhat V K, Bhan H L and Singh Y 1986 *J. Phys. A: Math. Gen.* **19** 3261–5
—— 1987 *J. Phys. A: Math. Gen.* **20** 4569–72
—— 1988 *J. Phys. A: Math. Gen.* **21** 3405–7
Cardy J L 1982 *J. Phys. A: Math. Gen.* **15** L593–5
Day A R and Lubensky T C 1982 *J. Phys. A: Math. Gen.* **15** L285–90
Delest M P and Dulucq S 1993 *Croatica Chemica Acta* **66** 59–80
Dhar D 1982 *Phys. Rev. Lett.* **49** 959-62
—— 1983 *Phys. Rev. Lett.* **51** 853–6
Dhar D, Phani M K and Barma M 1982 *J. Phys. A: Math. Gen.* **15** L279–84
Domb C and Hunter D L 1965 *Proc. Phys. Soc.* **86** 1147–51
Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
Fisher M E 1967 *Rep. Progr. Phys.* **30** 615–730
—— 1978 *Phys. Rev. Lett.* **40** 1610–3
Griffiths R B 1967 *Phys. Rev.* **158** 176–87
Joyce G S 1972 *Phase Transitions and Critical Phenomena* vol 2, ed C Domb and M S Green (London: Academic) pp 375–442
—— 1989 *J. Phys. A: Math. Gen.* **22** L919-24
Luke Y L 1969 *The Special Functions and their Approximations* vol 1 (New York: Academic)
Oberhettinger F 1965 *Handbook of Mathematical Functions* ed M Abramowitz and I A Stegun (New York: Dover) pp 555–66
Privman V and Forgacs G 1987 *J. Phys. A: Math. Gen.* **20** L543–7
Redner S and Yang Z R 1982 *J. Phys. A: Math. Gen.* **15** L177–87
Stanley H E, Redner S and Yang Z R 1982 *J. Phys. A: Math. Gen.* **15** L569–73
Whittaker E T and Watson G N 1965 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
Widom B 1965 *J. Chem. Phys.* **43** 3898–905
Wong R 1989 *Asymptotic Approximations of Integrals* (San Diego: Academic)